ITERATIVE APPROXIMATION METHOD FOR SOLVING HEAT CONDUCTION EQUATIONS

G. P. Kobranov

UDC 536.24

Convergence analysis is given for solving equations of heat conduction in the three-dimensional space on an electronic computer under involved boundary conditions; an iterative approximation method is used in which the sought function is determined by successive approximations with a subsequent approximation by Chebyshev polynomial of the same degree.

The use of electronic computers (EC) for numerical solution of a number of problems opens up new possibilities for developing methods thought not suitable for manual calculations. The iterative approximation method [1] belongs to this class. If we decide that to solve the equations of heat conduction the analysis of the advantages and disadvantages of this method compared with other numerical methods employed at present is beyond the scope of this article we shall consider the feasibility of solving the equation

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\varphi(x,\ y,\ z,\ \tau)=\Delta v+\varphi(x,\ y,\ z,\ \tau), \tag{1}$$

$$0<x<1;\ 0<y<1;\ 0<z<1;\ 0\leqslant\tau\leqslant 1$$

by means of the iterative approximation under the following boundary and initial conditions:

$$\begin{aligned}
&\frac{\partial v(0,\ y,\ z,\ \tau)}{\partial x}-b_{0x}v^4(0,\ y,\ z,\ \tau)=-\psi_{0x}(y,\ z,\ \tau);\\
&\frac{\partial v(1,\ y,\ z,\ \tau)}{\partial x}+b_{1x}v^4(1,\ y,\ z,\ \tau)=\psi_{1x}(y,\ z,\ \tau);\\
&\frac{\partial v(x,\ 0,\ z,\ \tau)}{\partial y}-b_{0y}v^4(x,\ 0,\ z,\ \tau)=-\psi_{0y}(x,\ z,\ \tau);\\
&\frac{\partial v(x,\ 1,\ z,\ \tau)}{\partial y}+b_{1y}v^4(x,\ 1,\ z,\ \tau)=\psi_{1y}(x,\ z,\ \tau);\\
&\frac{\partial v(x,\ y,\ 0,\ \tau)}{\partial z}-b_{0z}v^4(x,\ y,\ 0,\ \tau)=-\psi_{0z}(x,\ y,\ \tau);\\
&\frac{\partial v(x,\ y,\ 1,\ \tau)}{\partial z}+b_{1z}v^4(x,\ y,\ 1,\ \tau)=\psi_{1z}(x,\ y,\ \tau);\\
&v(x,\ y,\ z,\ 0)=\xi(x,\ y,\ z).
\end{aligned} \tag{2}$$

Equation (1) is now represented in the form

$$\frac{\partial P_{q,r,s,t}}{\partial \tau}=\frac{\partial^2 P_{q,r,s,t}}{\partial x^2}+\frac{\partial^2 P_{q,r,s,t}}{\partial y^2}+\frac{\partial^2 P_{q,r,s,t}}{\partial z^2}+\varphi+\Phi^*_{q,r,s,t}(x,\ y,\ z,\ \tau), \tag{3}$$

where

$$P_{q,r,s,t}=\sum_{i=0}^{q}\sum_{j=0}^{r}\sum_{k=0}^{s}\sum_{l=0}^{t}A_{ijkl}x^iy^jz^k\tau^l=\sum_{i,j,k,l}A_{ijkl}x^iy^jz^k\tau^l;\quad \max|\Phi^*_{q,r,s,t}|=E_{q,r,s,t}, \tag{4}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 3, pp. 513-523, September, 1974. Original article submitted July 5, 1972.

and its solution is sought by iterative approximation if the initial values of q, r, s, t are given by

$$v^{(p)}_{q,r,s,t} = v(x, y, z, 0) + \int_0^{\tau} \{\Delta P^{(p-1)}_{q,r,s,t} + \varphi\} d\tau;$$

$$0<x<1,\ 0<y<1,\ 0<z<1,\ 0\leqslant\tau\leqslant 1;$$

$$P^{(p-1)}_{q,r,s,t} = v^{(p-1)}_{q,r,s,t} + \Phi^{(p-1)}_{q,r,s,t};$$

$$0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1,\ 0\leqslant z\leqslant 1,\ 0\leqslant\tau\leqslant 1;$$

$$P^{(p-1)}_{q,r,s,t} = \sum_{i,j,k,l} a^{(p-1)}_{i,j,k,l} x^i y^j z^k \tau^l;$$

$$v^{(p)}_{q,r,s,t}(0, y, z, \tau) + \frac{b_{0x}}{n_x}[v^{(p)}_{q,r,s,t}(0, y, z, \tau)]^4 = P^{(p-1)}_{q,r,s,t}\Big|_{x=\frac{1}{n_x}} + \frac{\psi_{0x}(y, z, \tau)}{n_x};$$

$$v^{(p)}_{q,r,s,t}(1, y, z, \tau) + \frac{b_{1x}}{n_x}[v^{(p)}_{q,r,s,t}(1, y, z, \tau)]^4 = P^{(p-1)}_{q,r,s,t}\Big|_{x=1-\frac{1}{n_x}} + \frac{\psi_{1x}(y, z, \tau)}{n_x};$$

$$v^{(p)}_{q,r,s,t}(x, 0, z, \tau) + \frac{b_{0y}}{n_y}[v^{(p)}_{q,r,s,t}(x, 0, z, \tau)]^4 = P^{(p-1)}_{q,r,s,t}\Big|_{y=\frac{1}{n_y}} + \frac{\psi_{0y}(x, z, \tau)}{n_y}; \qquad (5)$$

$$v^{(p)}_{q,r,s,t}(x, 1, z, \tau) + \frac{b_{1y}}{n_y}[v^{(p)}_{q,r,s,t}(x, 1, z, \tau)]^4 = P^{(p-1)}_{q,r,s,t}\Big|_{y=1-\frac{1}{n_y}} + \frac{\psi_{1y}(x, z, \tau)}{n_y};$$

$$v^{(p)}_{q,r,s,t}(x, y, 0, \tau) + \frac{b_{0z}}{n_z}[v^{(p)}_{q,r,s,t}(x, y, 0, z)]^4 = P^{(p-1)}_{q,r,s,t}\Big|_{z=\frac{1}{n_z}} + \frac{\psi_{0z}(x, y, \tau)}{n_z};$$

$$v^{(p)}_{q,r,s,t}(x, y, 1, \tau) + \frac{b_{1z}}{n_z}[v^{(p)}_{q,r,s,t}(x, y, 1, z)]^4 = P^{(p-1)}_{q,r,s,t}\Big|_{z=1-\frac{1}{n_z}} + \frac{\psi_{1z}(x, y, \tau)}{n_z};$$

p = 1, 2, . . . , m is the ordinal number of the approximation; n_x, n_y, n_z denotes the number of steps in x, y, z respectively. For given q, r, s, t successive approximations are terminated when

$$\frac{|P^{(n_1-1)}_{q,r,s,t} - P^{(n_1)}_{q,r,s,t}|}{|P^{(n_1)}_{q,r,s,t}|} < \delta.$$

Increasing the degree of q, r, s, t by unity we now repeat the operations until the inequality

$$\frac{|P^{(n_2-1)}_{q+1,r+1,s+1,t+1} - P^{(n_2)}_{q+1,r+1,s+1,t+1}|}{|P^{(n_2)}_{q+1,r+1,s+1,t+1}|} < \delta,$$

is satisfied, δ denoting a suitably small value.

We evaluate now the quantity

$$\frac{|P^{(n_1)}_{q,r,s,t} - P^{(n_2)}_{q+1,r+1,s+1,t+1}|}{|P^{(n_2)}_{q+1,r+1,s+1,t+1}|} = \Delta_{q+1,r+1,s+1,t+1}. \qquad (6)$$

at the typical points of x, y, z, τ.

As soon as $\Delta_{q+1,r+1,s+1,t+1}$ we again increase by unity the degree of the polynomial and repeat the iterative approximation procedure until the condition

$$\Delta_{q+\xi,r+\eta,s+\vartheta,t+\chi} < \delta. \qquad (7)$$

has been reached.

The solution procedure is now analyzed in more detail.

The first approximation $v^{(1)}(x, y, z, \tau)$ is determined from

$$v^{(1)}(x, y, z, \tau) = v(x, y, z, 0) + \int_0^{\tau} \Delta v(x, y, z, 0)\, d\tau + \int_0^{\tau} \varphi\, d\tau, \qquad (8)$$

$$0<x<1,\ 0<y<1,\ 0<z<1,\ 0\leqslant\tau\leqslant 1,$$

$$v^{(1)}(0, y, z, \tau) + \frac{b_{0x}}{n_x}[v^{(1)}(0, y, z, \tau)]^4 = v^{(1)}\left(\frac{1}{n_x}, y, z, \tau\right) + \frac{\psi_{0x}(y, z, \tau)}{n_x},$$

$$v^{(1)}(1, y, z, \tau) + \frac{b_{1x}}{n_x}[v^{(1)}(1, y, z, \tau)]^4 = v^{(1)}\left(\left(1 - \frac{1}{n_x}\right), y, z, \tau\right) + \frac{\psi_{1x}(y, z, \tau)}{n_x};$$

$$v^{(1)}(x, 0, z, \tau) + \frac{b_{0y}}{n_y}[v^{(1)}(x, 0, z, \tau)]^4 = v^{(1)}\left(x, \frac{1}{n_y}, z, \tau\right) + \frac{\psi_{0y}(x, z, \tau)}{n_y}; \tag{9}$$

$$v^{(1)}(x, 1, z, \tau) + \frac{b_{1y}}{n_y}[v^{(1)}(x, 1, z, \tau)]^4 = v^{(1)}\left(x, \left(1 - \frac{1}{n_y}\right), z, \tau\right) + \frac{\psi_{1y}(x, z, \tau)}{n_y};$$

$$v^{(1)}(x, y, 0, \tau) + \frac{b_{0z}}{n_z}[v^{(1)}(x, y, 0, \tau)]^4 = v^{(1)}\left(x, y, \frac{1}{n_z}, \tau\right) + \frac{\psi_{0z}(x, y, \tau)}{n_z};$$

$$v^{(1)}(x, y, 1, \tau) + \frac{b_{1z}}{n_z}[v^{(1)}(x, y, 1, \tau)]^4 = v^{(1)}\left(x, y, \left(1 - \frac{1}{n_z}\right), \tau\right) + \frac{\psi_{1z}(x, y, \tau)}{n_z}.$$

Now $v^{(1)}(x, y, z, \tau)$ is approximated in the region $\{0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1,\ 0 \le \tau \le 1\}$ by the polynomial $P^{(1)}_{q, r, s, t}$, for example, in the following order.

By finding the function $v^{(1)}(x, 0, 0, 0)$ and approximating it on EC by the polynomial $\sum_{i=0}^{q} a_i^{(1)}(0, 0, 0)x^i$, one has

$$\sum_{i=0}^{q} a_i^{(1)}(0, 0, 0)\, x^i = v^{(1)}(x, 0, 0, 0) + \varepsilon_{(q)}^{(1)}(x, 0, 0, 0). \tag{10}$$

By carrying out a cycle of computations at the y nodes one obtains the system,

$$\begin{gathered}
\sum_{i=0}^{q} a_i^{(1)}\left(\frac{1}{n_y}, 0, 0\right) x^i = v^{(1)}\left(x, \frac{1}{n_y}, 0, 0\right) + \varepsilon_{(q)}^{(1)}\left(x, \frac{1}{n_y}, 0, 0\right);\\
\sum_{i=0}^{q} a_i^{(1)}\left(\frac{2}{n_y}, 0, 0\right) x^i = v^{(1)}\left(x, \frac{2}{n_y}, 0, 0\right) + \varepsilon_{(q)}^{(1)}\left(x, \frac{2}{n_y}, 0, 0\right);\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\sum_{i=0}^{q} a_i^{(1)}(1, 0, 0)\, x^i = v^{(1)}(x, 1, 0, 0) + \varepsilon_{(q)}^{(1)}(x, 1, 0, 0).
\end{gathered} \tag{11}$$

The functions $a_i^{(1)}(y, 0, 0)$, i = 0, 1, 2, . . . , q are approximated on the computer by the polynomials

$$\sum_{j=0}^{r} a_{ij}^{(1)}(0, 0)\, y^j = a_i^{(1)}(y, 0, 0) + \varepsilon_{i(r)}^{(1)}(y, 0, 0). \tag{12a}$$

By computing a cycle at the z nodes one obtains

$$\begin{gathered}
\sum_{j=0}^{r} a_{ij}^{(1)}\left(\frac{1}{n_z}, 0\right) y^j = a_i^{(1)}\left(y, \frac{1}{n_z}, 0\right) + \varepsilon_{i(r)}^{(1)}\left(y, \frac{1}{n_z}, 0\right);\\
\sum_{j=0}^{r} a_{ij}^{(1)}\left(\frac{2}{n_z}, 0\right) y^j = a_i^{(1)}\left(y, \frac{2}{n_z}, 0\right) + \varepsilon_{i(r)}^{(1)}\left(y, \frac{2}{n_z}, 0\right);\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\sum_{j=0}^{r} a_{ij}^{(1)}(1, 0)\, y^j = a_i^{(1)}(y, 1, 0) + \varepsilon_{i(r)}^{(1)}(y, 1, 0).
\end{gathered} \tag{12}$$

Then the expression $\varepsilon_{(q)}^{(1)}(x, y, z, 0)$ is also evaluated.

The functions $a_{ij}^{(1)}(z, 0)$, i = 0, 1, 2, . . . , q; j = 0, 1, 2, . . . , r are now approximated on the computer by the polynomials $\sum_{k=0}^{s} a_{ijk}^{(1)}(0)\, z^k$:

$$\sum_{k=0}^{s} a_{ijk}^{(1)}(0)\, z^k = a_{ij}^{(1)}(z,\ 0) + \varepsilon_{ij(s)}^{(1)}(z,\ 0). \tag{13}$$

Having concluded these computations at the τ nodes one obtains

$$\sum_{k=0}^{s} a_{ijk}^{(1)}\left(\frac{1}{n_\tau}\right) z^k = a_{ij}^{(1)}\left(z,\ \frac{1}{n_\tau}\right) + \varepsilon_{ij(s)}^{(1)}\left(z,\ \frac{1}{n_\tau}\right);$$

$$\sum_{k=0}^{s} a_{ijk}^{(1)}\left(\frac{2}{n_\tau}\right) z^k = a_{ij}^{(1)}\left(z,\ \frac{2}{n_\tau}\right) + \varepsilon_{ij(s)}^{(1)}\left(z,\ \frac{2}{n_\tau}\right); \tag{14}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{k=0}^{s} a_{ijk}^{(1)}(1)\, z^k = a_{ij}^{(1)}(z,\ 1) + \varepsilon_{ij(s)}^{(1)}(z,\ 1).$$

Then the quantities $\varepsilon_{(q)}^{(1)}(x, y, z, \tau)$ and $\varepsilon_{i(r)}^{(1)}(y, z, \tau)$ are also determined.

By approximating the functions $a_{ijk}^{(1)}(\tau)$ by polynomials one finds that

$$\sum_{l=0}^{t} a_{ijkl}^{(1)}\tau^l = a_{ijk}^{(1)}(\tau) + \varepsilon_{ijk(t)}^{(1)}(\tau), \tag{15}$$

$$i = 0,\ 1,\ 2,\ \ldots,\ q;\quad j = 0,\ 1,\ 2,\ \ldots,\ r;\quad k = 0,\ 1,\ 2,\ \ldots,\ s.$$

When this cycle of computations is terminated and by taking into account the relations of the type (10)-(15) one obtains

$$a_{ijk}^{(1)}(\tau) = \sum_{l=0}^{t} a_{ijkl}^{(1)}\tau^l - \varepsilon_{ijk(t)}^{(1)}(\tau);$$

$$a_{ij}^{(1)}(z,\ \tau) = \sum_{k=0}^{s} a_{ijk}^{(1)}(\tau)\, z^k - \varepsilon_{ij(s)}^{(1)}(z,\ \tau) = \sum_{k=0}^{s}\sum_{l=0}^{t} a_{ijkl}^{(1)} z^k \tau^l - \sum_{k=0}^{s}\varepsilon_{ijk}^{(1)}(\tau)\, z^k - \varepsilon_{ij(s)}^{(1)}(z,\ \tau);$$

$$a_{i}^{(1)}(y,\ z,\ \tau) = \sum_{j=0}^{r} a_{ij}^{(1)}(z,\ \tau)\, y^j - \varepsilon_{i(r)}^{(1)}(y,\ z,\ \tau) = \sum_{j=0}^{r}\sum_{k=0}^{s}\sum_{l=0}^{t} a_{ijkl}^{(1)} y^j z^k \tau^l - \sum_{j=0}^{r}\sum_{k=0}^{s}\varepsilon_{ijk(t)}^{(1)}(\tau)\, y^j z^k \tag{16}$$

$$-\sum_{j=0}^{r}\varepsilon_{ij(s)}^{(1)}(z,\ \tau)\, y^j - \varepsilon_{i(r)}^{(1)}(y,\ z,\ \tau);\quad v^{(1)} = P_{q,r,s,t}^{(1)} - \Phi_{q,r,s,t}^{(1)},$$

where

$$P_{q,r,s,t}^{(1)} = \sum_{i=0}^{q}\sum_{j=0}^{r}\sum_{k=0}^{s}\sum_{l=0}^{t} a_{ijkl}^{(1)} x^i y^j z^k \tau^l;$$

$$\Phi_{q,r,s,t}^{(1)} = \sum_{i=0}^{q}\sum_{j=0}^{r}\sum_{k=0}^{s}\varepsilon_{ijk(t)}^{(1)}(\tau)\, x^i y^j z^k + \sum_{i=0}^{q}\sum_{j=0}^{r}\varepsilon_{ij(s)}^{(1)}(z,\ \tau)\, x^i y^j + \sum_{i=0}^{q}\varepsilon_{i(r)}^{(1)}(y,\ z,\ \tau)\, x^i + \varepsilon_{(q)}^{(1)}(x,\ y,\ z,\ \tau).$$

The function $v_{q,r,s,t}^{(2)}$ can be obtained from the expressions

$$v_{q,r,s,t}^{(2)} = v(x,\ y,\ z,\ 0) + \int_0^\tau \Delta P_{q,r,s,t}^{(1)} d\tau + \int_0^\tau \varphi\, d\tau;$$

$$0 < x < 1,\ 0 < y < 1,\ 0 < z < 1,\ 0 \leqslant \tau \leqslant 1;$$

$$v_{q,r,s,t}^{(2)}(0,\ y,\ z,\ \tau) + \frac{b_{0x}}{n_x}\left[v_{q,r,s,t}^{(2)}(0,\ y,\ z,\ \tau)\right]^4 = P_{q,r,s,t}^{(1)}\Big|_{x=\frac{1}{n_x}} + \frac{\psi_{0x}(y,\ z,\ \tau)}{n_x}; \tag{17}$$

$$v_{q,r,s,t}^{(2)}(1,\ y,\ z,\ \tau) + \frac{b_{1x}}{n_x}\left[v_{q,r,s,t}^{(2)}(1,\ y,\ z,\ \tau)\right]^4 = P_{q,r,s,t}^{(1)}\Big|_{x=1-\frac{1}{n_x}} + \frac{\psi_{1x}(y,\ z,\ \tau)}{n_x};$$

$$v^{(2)}_{q,r,s,t}(x,\ 0,\ z,\ \tau)+\frac{b_{0y}}{n_y}[v^{(2)}_{q,r,s,t}(x,\ 0,\ z,\ \tau)]^4=P^{(1)}_{q,r,s,t}\Big|_{y=\frac{1}{n_y}}+\frac{\psi_{0y}(x,\ z,\ \tau)}{n_y};$$

$$v^{(2)}_{q,r,s,t}(x,\ 1,\ z,\ \tau)+\frac{b_{1y}}{n_y}[v^{(2)}_{q,r,s,t}(x,\ 1,\ z,\ \tau)]^4=P^{(1)}_{q,r,s,t}\Big|_{y=1-\frac{1}{n_y}}+\frac{\psi_{1y}(x,\ z,\ \tau)}{n_y};\qquad(18)$$

$$v^{(2)}_{q,r,s,t}(x,\ y,\ 0,\ \tau)+\frac{b_{0z}}{n_z}[v^{(2)}_{q,r,s,t}(x,\ y,\ 0,\ \tau)]^4=P^{(1)}_{q,r,s,t}\Big|_{z=\frac{1}{n_z}}+\frac{\psi_{0z}(x,\ y,\ \tau)}{n_z};$$

$$v^{(2)}_{q,r,s,t}(x,\ y,\ 1,\ \tau)+\frac{b_{1z}}{n_z}[v^{(2)}_{q,r,s,t}(x,\ y,\ 1,\ \tau)]^4=P^{(1)}_{q,r,s,t}\Big|_{z=1-\frac{1}{n_z}}+\frac{\psi_{1z}(x,\ y,\ \tau)}{n_z};$$

$$0\leqslant\tau\leqslant1;\ v(x,\ y,\ z,\ 0)=\xi(x,\ y,\ z).$$

The function $v^{(2)}_{q,r,s,t}(x, y, z, \tau)$ is now approximated by means of

$$v^{(2)}_{q,r,s,t}=P^{(2)}_{q,r,s,t}-\Phi^{(2)}_{q,r,s,t},\qquad(19)$$

where

$$P^{(2)}_{q,r,s,t}=\sum_{i,j,k,l}a^{(2)}_{ijkl}x^iy^jz^k\tau^l;$$

$$\Phi^{(2)}_{q,r,s,t}=\sum_{i,j,k}\varepsilon^{(2)}_{ijk(t)}x^iy^jz^k+\sum_{i,j}\varepsilon^{(2)}_{ij(s)}x^iy^j+\sum_{i}\varepsilon^{(2)}_{i(r)}x^i+\varepsilon^{(2)}_{(q)};\qquad(20)$$

$$i=0,\ 1,\ 2,\ \ldots,\ q;\ j=0,\ 1,\ 2,\ \ldots,\ r;\ k=0,\ 1,\ 2,\ \ldots,\ s;$$
$$l=0,\ 1,\ 2,\ \ldots,\ t;\ 0\leqslant x\leqslant1;\ 0\leqslant y\leqslant1;\ 0\leqslant z\leqslant1;\ 0\leqslant\tau\leqslant1.$$

Proceeding further in this way the relations of the form (17)-(20) are obtained for the m-th approximation in which the subscripts 1, 2 are replaced by m-1, m.

A convergence analysis is now carried out for the solution of Eq. (3). To this end the limit is found of the operator for $m\to\infty$,

$$L^{(m)}_{q,r,s,t}=\frac{\partial P^{(m)}_{q,r,s,t}}{\partial\tau}-\Delta P^{(m)}_{q,r,s,t}-\varphi(x,\ y,\ z,\ \tau).\qquad(21)$$

By taking into account (17) and (19) one finds

$$\frac{\partial P^{(m)}_{q,r,s,t}}{\partial\tau}=\Delta P^{(m-1)}_{q,r,s,t}+\varphi(x,\ y,\ z,\ \tau)+\frac{\partial\Phi^{(m)}_{q,r,s,t}}{\partial\tau},\qquad(22)$$

and hence

$$L^{(m)}_{q,r,s,t}=(\Delta P^{(m-1)}_{q,r,s,t}-\Delta P^{(m)}_{q,r,s,t})+\frac{\partial\Phi^{(m)}_{q,r,s,t}}{\partial\tau},\qquad(23)$$
$$m=2,\ 3,\ \ldots,\ n.$$

It follows from the expressions of the type (16) and (19) that

$$P^{(m-1)}_{q,r,s,t}-P^{(m)}_{q,r,s,t}=(v^{(m-1)}_{q,r,s,t}-v^{(m)}_{q,r,s,t})+(\Phi^{(m-1)}_{q,r,s,t}-\Phi^{(m)}_{q,r,s,t});$$
$$m=2,\ 3,\ \ldots,\ n.$$

Since

$$v^{(m)}_{q,r,s,t}=v(x,\ y,\ z,\ 0)+\int_0^\tau\Delta P^{(m-1)}_{q,r,s,t}\,d\tau+\int_0^\tau\varphi(x,\ y,\ z,\ \tau)\,d\tau,$$

therefore

$$P^{(m-1)}_{q,r,s,t}-P^{(m)}_{q,r,s,t}=\int_0^\tau(\Delta P^{(m-2)}_{q,r,s,t}-\Delta P^{(m-1)}_{q,r,s,t})\,d\tau+\Phi^{(m-1,m)}_{q,r,s,t},$$

where

$$\Phi^{(m-1,m)}_{q,r,s,t}=\Phi^{(m-1)}_{q,r,s,t}-\Phi^{(m)}_{q,r,s,t}.$$

An assumption is now made that the function v(x, y, z, 0) is bounded and that the derivative $d(\Delta v)/d\tau$ exists in the entire region in which the function v is determined.

One can obtain

$$|P^{(1)}_{q,r,s,t}-P^{(2)}_{q,r,s,t}|\leqslant M^{(1)}+|\overline{\Phi}^{(1,2)}_{q,r,s,t}|_{\max},$$
$$|\Delta P^{(m-1)}_{q,r,s,t}-\Delta P^{(m)}_{q,r,s,t}|\leqslant N^{(m-1)}|P^{(m-1)}_{q,r,s,t}-P^{(m)}_{q,r,s,t}|, \quad (24)$$

where

$$M^{(1)}=M+\max\left|\int_0^\tau \Delta\Phi^{(1)}_{q,r,s,t}\,d\tau\right|;$$
$$|\overline{\Phi}^{(1,2)}_{q,r,s,t}|_{\max}=(|\Phi^{(1)}_{q,r,s,t}|+|\Phi^{(2)}_{q,r,s,t}|)_{\max};$$

and M, $N^{(m-1)}$ are finite values.

One now finds $L^{(p)}_{q,r,s,t}$ (p = 1, 2, . . . , m) bearing in mind the inequalities (24):

$$L^{(1)}_{q,r,s,t}=\frac{\partial P^{(1)}_{q,r,s,t}}{\partial\tau}-\Delta P^{(1)}_{q,r,s,t}-\varphi(x,y,z,\tau)\leqslant|\Delta\Phi^{(1)}_{q,r,s,t}|_{\max}+\left|\frac{\partial\Phi^{(1)}_{q,r,s,t}}{\partial\tau}\right|+M_0;$$

$$L^{(4)}_{q,r,s,t}\leqslant N^{(1)}N^{(2)}N^{(3)}(M^{(1)}+|\overline{\Phi}^{(1,2)}_{q,r,s,t}|_{\max})\frac{\tau^2}{2!}+N^{(2)}N^{(3)}|\overline{\Phi}^{(2,3)}_{q,r,s,t}|_{\max}\tau+N^{(3)}|\overline{\Phi}^{(3,4)}_{q,r,s,t}|_{\max}+\left|\frac{\partial\Phi^{(4)}_{q,r,s,t}}{\partial\tau}\right|;$$

$$L^{(2)}_{q,r,s,t}=(\Delta P^{(1)}_{q,r,s,t}-\Delta P^{(2)}_{q,r,s,t})+\frac{\partial\Phi^{(2)}_{q,r,s,t}}{\partial\tau}\leqslant N^{(1)}(M^{(1)}+|\overline{\Phi}^{(1,2)}_{q,r,s,t}|_{\max})+\left|\frac{\partial\Phi^{(2)}_{q,r,s,t}}{\partial\tau}\right|;$$

$$L^{(3)}_{q,r,s,t}=(\Delta P^{(2)}_{q,r,s,t}-\Delta P^{(3)}_{q,r,s,t})+\frac{\partial\Phi^{(3)}_{q,r,s,t}}{\partial\tau}\leqslant N^{(2)}\int_0^\tau N^{(1)}|P^{(1)}_{q,r,s,t}-P^{(2)}_{q,r,s,t}|\,d\tau+N^{(2)}|\overline{\Phi}^{(2,3)}_{q,r,s,t}|_{\max}$$
$$+\left|\frac{\partial\Phi^{(3)}_{q,r,s,t}}{\partial\tau}\right|\leqslant N^{(1)}N^{(2)}(M^{(1)}+|\overline{\Phi}^{(1,2)}_{q,r,s,t}|_{\max})\tau+N^{(2)}|\overline{\Phi}^{(2,3)}_{q,r,s,t}|_{\max}+\left|\frac{\partial\Phi^{(3)}_{q,r,s,t}}{\partial\tau}\right|;$$

. . . .

$$L^{(m)}_{q,r,s,t}\leqslant N^{(1)}N^{(2)}\ldots N^{(m-1)}(M^{(1)}+|\overline{\Phi}^{(1,2)}_{q,r,s,t}|_{\max})\frac{\tau^{m-2}}{(m-2)!}$$
$$+N^{(2)}N^{(3)}\ldots N^{(m-1)}|\overline{\Phi}^{(2,3)}_{q,r,s,t}|_{\max}\frac{\tau^{m-3}}{(m-3)!}+\ldots \quad (25)$$
$$\ldots+N^{(m-2)}N^{(m-1)}|\overline{\Phi}^{(m-2,m-1)}_{q,r,s,t}|_{\max}\tau+N^{(m-1)}|\overline{\Phi}^{(m-i,m)}_{q,r,s,t}|_{\max}+\left|\frac{\partial\Phi^{(m)}_{q,r,s,t}}{\partial\tau}\right|.$$

The series on the right of the inequality (25) converges for $m\to\infty$. Consequently,

$$\lim_{m\to\infty}L^{(m)}_{q,r,s,t}=L_{q,r,s,t};$$

$$\lim_{m\to\infty}\frac{\partial P^{(m)}_{q,r,s,t}}{\partial\tau}-\lim_{m\to\infty}\Delta P^{(m)}_{q,r,s,t}-\varphi(x,y,z,\tau)=\frac{\partial P_{q,r,s,t}}{\partial\tau}-\Delta P_{q,r,s,t}-\varphi(x,y,z,\tau);$$

$$\lim_{m\to\infty}a^{(m)}_{ijkl}=A_{ijkl};$$
$$i=0,1,2,\ldots,q;$$
$$j=0,1,2,\ldots,r;\quad k=0,1,2,\ldots,s;\quad l=0,1,2,\ldots,t; \quad (26)$$
$$\lim_{m\to\infty}P^{(m)}_{q,r,s,t}=P_{q,r,s,t}.$$

Proceeding to the limit in the expression (22) one obtains

$$\lim_{m\to\infty}\frac{\partial\Phi^{(m)}_{q,r,s,t}}{\partial\tau}=\frac{\partial\Phi_{q,r,s,t}}{\partial\tau},$$

$$\lim_{m\to\infty}\frac{\partial\varepsilon^{(m)}_{ijk(t)}}{\partial\tau}=\frac{\partial\varepsilon_{ijk(t)}}{\partial\tau};\quad \lim_{m\to\infty}\frac{\partial\varepsilon^{(m)}_{ij(s)}}{\partial\tau}=\frac{\partial\varepsilon_{ij(s)}}{\partial\tau}; \quad (27)$$

$$\lim_{m\to\infty}\frac{\partial\varepsilon_{i(r)}^{(m)}}{\partial\tau}=\frac{\partial\varepsilon_{i(r)}}{\partial\tau};\quad \lim_{m\to\infty}\varepsilon_{ijk(t)}^{(m)}=\varepsilon_{ijk(t)};$$

$$\lim_{m\to\infty}\varepsilon_{ij(s)}^{(m)}=\varepsilon_{ij(s)};\ \lim_{m\to\infty}\varepsilon_{i(r)}^{(m)}=\varepsilon_{i(r)};$$

$$\lim_{m\to\infty}\Phi_{q,r,s,t}^{(m)}=\Phi_{q,r,s,t}\,. \tag{28}$$

In view of (27) one obtains from the expression (23) that

$$\lim_{m\to\infty}\Delta P_{q,r,s,t}^{(m-1)}=L_{q,r,s,t}+\Delta P_{q,r,s,t}-\frac{\partial\Phi_{q,r,s,t}}{\partial\tau}=\Delta P_{q,r,s,t}\,.$$

Hence,

$$L_{q,r,s,t}\leqslant\left|\frac{\partial\Phi_{q,r,s,t}}{\partial\tau}\right|;$$

$$0<x<1;\ 0<y<1;\ 0<z<1;\ 0\leqslant\tau\leqslant 1.$$

Increasing now by unity the degree of the polynomials in q, r, s, t which approximate the function $v(x, y, z, \tau)$ one finds that

$$L_{q+1,\,r+1,\,s+1,\,t+1}\leqslant L_{q,r,s,t};\ \ \Phi_{q+1,\,r+1,\,s+1,\,t+1}\leqslant\Phi_{q,r,s,t}\,.$$

Finally, when the degree of the polynomials becomes infinite the errors and the operator of the form as in (21) approach zero and the expression (4) is the exact solution of (1).

The convergence analysis of the solution is now carried out when the boundary conditions are given by (5). To this end with $m\to\infty$ it is necessary to find the limits of the operators

$$L_{q,r,s,t\;x=0}^{(m)}=v_{q,r,s,t}^{(m)}(0,\ y,\ z,\ \tau)+\frac{b_{0x}}{n_x}\,[v_{q,r,s,t}^{(m)}(0,\ y,\ z,\ \tau)]^4-P_{q,r,s,t}^{(m)}\big|_{x=\frac{1}{n_x}};$$

$$L_{q,r,s,t,x=1}^{(m)}=v_{q,r,s,t}^{(m)}(1,\ y,\ z,\ \tau)+\frac{b_{1x}}{n_x}\,[v_{q,r,s,t}^{(m)}(1,\ y,\ z,\ \tau)]^4-P_{q,r,s,t}^{(m)}\big|_{x=1-\frac{1}{n_x}};$$

$$L_{q,r,s,t,\ y=0}^{(m)}=v_{q,r,s,t}^{(m)}(x,\ 0,\ z,\ \tau)+\frac{b_{0y}}{n_y}\,[v_{q,r,s,t}^{(m)}(x,\ 0,\ z,\ \tau)]^4-P_{q,r,s,t}^{(m)}\big|_{y=\frac{1}{n_y}}; \tag{29}$$

$$L_{q,r,s,t,y=1}^{(m)}=v_{q,r,s,t}^{(m)}(x,\ 1,\ z,\ \tau)+$$
$$+\frac{b_{1y}}{n_y}\,[v_{q,r,s,t}^{(m)}(x,\ 1,\ z,\ \tau)]^4-P_{q,r,s,t}^{(m)}\big|_{y=1-\frac{1}{n_y}};$$

$$L_{q,r,s,t,z=0}^{(m)}=v_{q,r,s,t}^{(m)}(x,\ y,\ 0,\ \tau)+$$
$$+\frac{b_{0z}}{n_z}\,[v_{q,r,s,t}^{(m)}(x,\ y,\ 0,\ \tau)]^4-P_{q,r,s,t}^{(m)}\big|_{z=\frac{1}{n_z}};$$

$$L_{q,r,s,t,z=1}^{(m)}=v_{q,r,s,t}^{(m)}(x,\ y,\ 1,\ \tau)+$$
$$+\frac{b_{1z}}{n_z}\,[v_{q,r,s,t}^{(m)}(x,\ y,\ 1,\ \tau)]^4-P_{q,r,s,t}^{(m)}\big|_{z=1-\frac{1}{n_z}}\,.$$

The relations (5) yield

$$v_{q,r,s,t}^{(m)}(0,\ y,\ z,\ \tau)=f\left(P_{q,r,s,t}^{(m-1)}\big|_{x=\frac{1}{n_x}},\ \frac{\psi_{0x}}{n_x}\right);$$

$$v_{q,r,s,t}^{(m)}(1,\ y,\ z,\ \tau)=f\left(P_{q,r,s,t}^{(m-1)}\big|_{x=1-\frac{1}{n_x}},\ \frac{\psi_{1x}}{n_x}\right);$$

$$v_{q,r,s,t}^{(m)}(x,\ 0,\ z,\ \tau)=f\left(P_{q,r,s,t}^{(m-1)}\big|_{y=\frac{1}{n_y}},\ \frac{\psi_{0y}}{n_y}\right); \tag{30}$$

$$v_{q,r,s,t}^{(m)}(x,\ 1,\ z,\ \tau)=f\left(P_{q,r,s,t}^{(m-1)}\big|_{y=1-\frac{1}{n_y}},\ \frac{\psi_{1y}}{n_y}\right);$$

$$v^{(m)}_{q,r,s,t}(x,\ y,\ 0,\ \tau)=f\left(P^{(m-1)}_{q,r,s,t}\Big|_{z=\frac{1}{n_z}},\ \frac{\psi_{0z}}{n_z}\right);$$

$$v^{(m)}_{q,r,s,t}(x,\ y,\ 1,\ \tau)=f\left(P^{(m-1)}_{q,r,s,t}\Big|_{z=1-\frac{1}{n_z}},\ \frac{\psi_{1z}}{n_z}\right);$$

By using (26) it is not difficult to prove that

$$\lim_{m\to\infty} f\left(P^{(m-1)}_{q,r,s,t}\Big|_{x=\frac{1}{n_x}},\ \frac{\psi_{0x}}{n_x}\right)=f\left(P_{q,r,s,t}\Big|_{x=\frac{1}{n_x}},\ \frac{\psi_{0x}}{n_x}\right);$$

$$\lim_{m\to\infty} f\left(P^{(m-1)}_{q,r,s,t}\Big|_{x=1-\frac{1}{n_x}},\ \frac{\psi_{1x}}{n_x}\right)=f\left(P_{q,r,s,t}\Big|_{x=1-\frac{1}{n_x}},\ \frac{\psi_{1x}}{n_x}\right);$$

$$\lim_{m\to\infty} f\left(P^{(m-1)}_{q,r,s,t}\Big|_{y=\frac{1}{n_y}},\ \frac{\psi_{0y}}{n_y}\right)=f\left(P_{q,r,s,t}\Big|_{y=\frac{1}{n_y}},\ \frac{\psi_{0y}}{n_y}\right);$$

$$\lim_{m\to\infty} f\left(P^{(m-1)}_{q,r,s,t}\Big|_{y=1-\frac{1}{n_y}},\ \frac{\psi_{1y}}{n_y}\right)=f\left(P_{q,r,s,t}\Big|_{y=1-\frac{1}{n_y}},\ \frac{\psi_{1y}}{n_y}\right);$$

$$\lim_{m\to\infty} f\left(P^{(m-1)}_{q,r,s,t}\Big|_{z=\frac{1}{n_z}},\ \frac{\psi_{0z}}{n_z}\right)=f\left(P_{q,r,s,t}\Big|_{z=\frac{1}{n_z}},\ \frac{\psi_{0z}}{n_z}\right);$$

$$\lim_{m\to\infty} f\left(P^{(m-1)}_{q,r,s,t}\Big|_{z=1-\frac{1}{n_z}},\ \frac{\psi_{1z}}{n_z}\right)=f\left(P_{q,r,s,t}\Big|_{z=1-\frac{1}{n_z}},\ \frac{\psi_{1z}}{n_z}\right).$$

Then together with (30) one also obtains

$$\lim_{m\to\infty} v^{(m)}_{q,r,s,t}\ (0,\ y,\ z,\ \tau)=v_{q,r,s,t}(0,\ y,\ z,\ \tau);$$

$$\lim_{m\to\infty} v^{(m)}_{(q,r,s,t}(1,\ y,\ z,\ \tau)=v_{q,r,s,t}(1,\ y,\ z,\ \tau);$$

$$\lim_{m\to\infty} v^{(m)}_{q,r,s,t}(x,\ 0,\ z,\ \tau)=v_{q,r,s,t}(x,\ 0,\ z,\ \tau);$$

$$\lim_{m\to\infty} v^{(m)}_{q,r,s,t}(x,\ 1,\ z,\ \tau)=v_{q,r,s,t}(x,\ 1,\ z,\ \tau);$$

$$\lim_{m\to\infty} v^{(m)}_{q,r,s,t}(x,\ y,\ 0,\ \tau)=v_{q,r,s,t}(x,\ y,\ z,\ \tau);$$

$$\lim_{m\to\infty} v^{(m)}_{q,r,s,t}(x,\ y,\ 1,\ \tau)=v_{q,r,s,t}(x,\ y,\ 1,\ \tau).$$

Consequently,

$$\lim_{m\to\infty} L^{(m)}_{q,r,s,t\ x=0}=L_{q,r,s,t,x=0};\quad \lim_{m\to\infty} L^{(m)}_{q,r,s,t,x=1}=L_{q,r,s,t,x=1};$$

$$\lim_{m\to\infty} L^{(m)}_{q,r,s,t,y=0}=L_{q,r,s,t,y=0};\quad \lim_{m\to\infty} L^{(m)}_{q,r,s,t,y=1}=L_{q,r,s,t,y=1};$$

$$\lim_{m\to\infty} L^{(m)}_{q,r,s,t,z=0}=L_{q,r,s,t,z=0};\quad \lim_{m\to\infty} L^{(m)}_{q,r,s,t,z=1}=L_{q,r,s,t,z=1}. \tag{31}$$

The procedure is convergent.

In the limit for infinitely high degree of the polynomials the operators (31) tend to the functions on the right of (2).

The author is sincerely grateful to I. V. Fryazinov for the trouble he has taken to familiarize himself with the essential material of this article and for his valuable critical remarks.

LITERATURE CITED

1. G. P. Korbanov, "Aerodynamics of rarified gases," Symp. 5. Izd. Zhdanov State Univ., Leningrad (1971), pp. 119-132.